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Gabriel Cruz, Marcos Fabian

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Double Trouble: How Tradable Permits Address Externalities and Inequality

Aldo Gutiérrez-Mendieta[1]
aldogm@tec.mx
Department of Economics, Tecnológico de Monterrey, Mexico City, Mexico
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Abstract: This paper examines how a market for tradable consumption permits can address two persistent failures in environmental and public economics: negative externalities generated by polluting goods and limited access to consumption opportunities driven by income inequality. Consumers face a choice between two otherwise identical goods, a green good and a brown good, which they may consume in any combination. The only differences are that brown goods generate pollution and degrade environmental quality, which is a public good, while green goods avoid this harm but are more expensive. Building on the literature of individual transferable quotas (ITQs) and impure public goods, I propose a system in which consumers must hold permits to purchase brown goods. These permits can be traded in a competitive market, with prices adjusting to equate demand and supply. The regulator sets the initial allocation of permits, which in turn affects both environmental outcomes and the income distribution in the economy. The analysis shows that, under general conditions, a competitive equilibrium in this permit market exists, and its efficiency and distributional properties depend critically on the initial allocation of permits. When all consumers participate in this market, whether as buyers or sellers, there exists an equilibrium price that equalizes the marginal rates of substitution across participants and achieves the socially optimal outcome. In cases where some consumers are constrained or abstain from trading, the equilibrium may fall short of the first best, yet it still improves welfare by reducing pollution and redistributing income. Key results include conditions for the existence of equilibrium, comparative statics of permit prices with respect to initial allocations, and criteria for identifying when the optimal price has been reached to guide regulatory policy. Even if imperfect, such a permit market offers a flexible, decentralized mechanism that reduces environmental damage while providing redistribution to low-income consumers. Rather than replacing direct regulation or traditional redistribution, this approach complements them by aligning private incentives with broader social objectives.

Keywords: Environmental Quality, Tradable Permits, Environmental Regulation, Externalities, Income Inequality, Individual Transferable Quotas (ITQs)

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1. : Introduction

Addressing climate change, environmental degradation, and income inequality ranks among the most urgent global priorities (IPCC 2007, UN 2022, World Bank 2022). Climate-related impacts disproportionately affect vulnerable populations and threaten efforts to reduce poverty. In response, the promotion of sustainable consumption—particularly through the adoption of environmentally friendly or "green" goods—has emerged as a potential strategy. These goods offer environmental and health benefits and may help decouple economic growth from environmental harm. However, their higher cost often limits access for low-income households, raising concerns about the fairness and feasibility of a green transition, particularly in societies with high levels of inequality.

While income inequality has been extensively studied since the 1990s (e.g., Banerjee and Piketty 2005; Piketty 2015; Rey and Sastré Gutiérrez 2015; Saez and Zucman 2016; Goda, Onaran and Stockhammer 2017; Dong and Hao 2018; Piketty, Yang and Zucman 2019), its connection to environmental quality remains debated. Some studies argue that income and power inequalities contribute to environmental problems (Boyce 1994; Torras and Boyce 1998; Downey 2015; Uzar and Eyuboglu 2019), while others find no clear relationship or even suggest a positive effect on environmental quality as income inequality increases (Scruggs 1998; Heerink, Mulatu, and Bulte 2001; Grunewald, Klasen and Martínez-Zarzoso 2017).These mixed findings indicate a need for deeper understanding of how inequality shapes environmental behavior and outcomes.

This paper contributes to this discussion by examining how consumers choose between environmentally friendly "green goods" and cheaper, more polluting "brown goods." I introduce a baseline model in which individuals can choose either option or both. Brown goods are characterized by their lower cost and the negative environmental externalities they generate. I use this model to explore a policy intervention that addresses both the environmental harms associated with brown goods and the broader issue of income inequality. A key insight from the model is that low-income consumers may rely on brown goods not out of preference, but because of financial constraints. Their consumption patterns are shaped less by environmental indifference and more by economic necessity. Therefore, an effective policy must address both environmental and equity concerns simultaneously.

To this end, I propose a market-based intervention: a system of tradable consumption permits for brown goods. Inspired by the literature on individual transferable quotas (ITQs), which have demonstrated efficiency with minimal information requirements (Arnason 1990, Heaps 2003, Arnason 2009), the policy would cap total brown goods consumption by issuing a fixed number of permits to consumers. Once their allocation is exhausted, consumers would need to purchase additional permits from others who have unused ones.

This trading system allows individuals who consume fewer brown —typically those with lower incomes—to sell their excess permits, generating supplemental income. These earnings could then be used to access green goods or satisfy other consumption needs. In doing so, the policy enables a form of decentralized income redistribution while maintaining environmental goals. Therefore, the primary objective of this policy to internalize the environmental costs of brown goods by raising their effective price and capping total consumption. This ensures a minimum level of environmental quality beyond what a competitive market would produce. At the same time, the permit market facilitates income transfers to financially constrained consumers, potentially improving both environmental outcomes and social equity.

Based on the results of the model, if all consumers act as price-takers and voluntarily participate in the permit market, the initial allocation of permits can be used as a policy tool to influence the market-clearing price. This price would equalize the marginal rates of substitution across individuals, leading to an efficient level of environmental quality and a reduction in income inequality.

This paper also contributes to the existing literature on impure public goods provision (Cornes and Sandler 1984, 1994; Kotchen 2005, 2006, 2009) by proposing an optimal policy framework for green and brown goods consumption. Previous approaches, such as Pigouvian taxes and direct transfers, face challenges when consumers have heterogeneous preferences, necessitating individualized tax schemes (Atkinson and Stern 1974; Boadway and Keen 1993; Itaya, de Meza, and Myles 1997). To overcome this issue, alternative methods have been explored, including compensatory taxation (McMahon 2015), incentive-compatible mechanisms (Wichman 2016), and cost-sharing mechanisms (Chan and Dinelli 2020). However, these approaches generally rely on detailed information about individual preferences or the ability to implement personalized pricing, which may be unrealistic in practice. In contrast, the permit system I propose avoids these informational and implementation challenges. By allowing consumers to trade permits, the policy leverages decentralized market behavior to determine an equilibrium price, aligning incentives without requiring personalized interventions.

Nonetheless, the effectiveness of this approach depends on full participation in the permit market—an ideal that may not hold in real-world settings. If some consumers opt out of the market, differences in marginal rates of substitution may persist across individuals, potentially undermining the achievement of a socially optimal level of environmental quality. Even so, the proposed permit system offers a practical and robust second-best solution. It advances both environmental and equity objectives, even under imperfect conditions, by reducing inequality and encouraging more sustainable consumption patterns.

The rest of this paper is organized as follows. In section 2, I review the literature on Individual Transferable Quotas (ITQs), tradable permits, and their use as policy instruments. In Section 3, I present the baseline model alongside the proposed permit system model, emphasizing the differences in decision-making processes between buyers and sellers. Furthermore, I also outline the outcomes that result from the market equilibrium. Finally, in Section 4, I conclude with a discussion of the main findings.

2. : Literature Review

The concept of using tradable permits to address pollution and manage common-pool resources emerged from the economic discussions of the 1960s and 1970s, focused on the theory of externalities (Dales, 1968; Baumol and Oates, 1988). It is based on the recognition that environmental issues arise due to the absence of well-functioning markets (Gordon, 1954; Coase, 1960; Hardin, 1968). By implementing tradeable permits, the aim is to establish a market that enforces property rights related to environmental usage, fostering a competitive market with minimal transaction costs. This enables individuals to negotiate and trade these rights efficiently.

Tradable permit schemes have the potential to effectively address challenges such as overfishing and pollution without relying on direct regulatory measures like taxes or technical standards, while maximizing economic rents (Arnason, 1990; Shotton, 2000). Essential characteristics of these permits include permanence, security, exclusivity, and tradability (Scott 1989; Grafton 1996; Squires, Kirkley, and Tisdell 1995; Arnason, 2007; Libecap 2007). Ultimately, the objective is to minimize the costs associated with achieving predetermined environmental targets.

Examples of this approach include the implementation of individual transferable quotas (ITQ) in fisheries, which have been studied extensively (McCay, Creed, Finlayson, Apostle and Mikalsen 1995; Hermann 2000; Arnason 2002, 2005; Hilbron, Orensanz and Parma 2005; Stewart, Walshe, & Moodie 2006; Chu, 2009; Brinson and Thunberg 2016), and the adoption of emission trading schemes (ETS) to mitigate pollutant discharge into the environment (Demailly and Quirion 2008; Cullenward and Coghlan 2016; Martin, Muûls, and Wagner 2016; Narassimhan, Gallagher, Koester and Alejo 2018). In practice, specifically for ETS, a central regulator assigns or sells a limited number of emission permits based on an emission reduction target. Polluters are obligated to hold permits equivalent to or exceeding their emitted quantity. This system resembles a cap-and-trade mechanism that employs quantity as a planning instrument to achieve a desired output level, such as clean air, while motivating producers to minimize production costs (Weitzman, 1974).

Although permit schemes have been effective in enhancing the efficiency of environmental resource management, particularly in U.S. fisheries (Wang, 1995; Annala 1996; Weninger, 1998; Arnason 2012; Brinson and Thunberg, 2016), they have faced criticism for their insensitivity to social needs, inequitable distributional consequences, and disproportionate impact on small stakeholders (Copes and Charles 2004; Olson 2011; Knapp 2011; Carothers and Chambers 2012; Soliman 2014). Therefore, it is crucial to consider the distributional effects of initial permit allocation to ensure access for income-constrained individuals while balancing equity and efficiency concerns (Laffont and Robert 1996; Che, Gale and Kim 2013; Holzer and McConnell 2023). This chapter contributes to the existing literature by proposing permit allocation as a policy tool that favors income-constrained consumers who can derive greater monetary gains from reselling permits, while also ensuring that their polluting consumption remains restrained due to their economic disadvantages.

3. The model

3.1 The Baseline Model

To address the research questions, I begin by developing a baseline model of a small economy consisting of $n > 1$ consumers, based on the impure public model (Cornes and Sandler 1984, 1994) and the environmentally friendly consumption model (Kotchen 2005, 2006). Each consumer has an initial endowment w_i that is exogenously given. Preferences are defined over two components: total private consumption (x_i) and environmental quality (Y). The variable x_i functions as a standard private good, while Y is affected by the collective actions of all the consumers. As such, it is possible to think of Y as a public good, non-rival and non-excludable in nature. There are no markets for either x_i or Y. Instead, consumers allocate their endowment between two consumption options: green goods (g_i) and brown goods (b_i). I assume constant returns to scale in the production of both goods, which are sold in perfectly competitive markets. The price per unit of g_i is greater than the price per unit of b_i, such that $P_g > P_b$.

Each unit of g_i contributes proportionally to x_i but has no effect on environmental quality Y. In contrast, each unit of b_i provides the same benefit in terms of x_i as g_i, but also imposes a negative externality on Y. Let $\sum_{i=1}^{n} b_i$ denote the total consumption of brown goods in the economy. The level of environmental quality is then defined as $Y = Y^0 - \sum_{i=1}^{n} b_i$, where Y^0 is a positive and exogenous constant representing the baseline level of environmental quality in the absence of pollution.

Each consumer maximizes utility subject to their budget constraint, taking prices as given. The individual utility maximization problem is as follows:

$$\max_{x_i, Y, g_i, b_i} \left\{ U_i(x_i, Y) \middle| \begin{array}{l} w_i = P_g g_i + P_b b_i \\ x_i = g_i + b_i \\ Y = Y^0 - \sum_{i=1}^{n} b_i \\ g_i \geq 0,\ b_i \geq 0 \end{array} \right\}$$

I assume that each individual utility function $U_i(x_i, Y)$ is strictly increasing in both arguments and strictly quasiconcave. These assumptions ensure that the utility maximization problem has a unique solution, and that consumers choose their optimal bundle of x_i and Y. However, since both green and brown goods contribute equally to x_i, utility maximization does not directly determine the consumption of g_i and b_i, but rather the total level of private consumption. As a result, it is possible for some consumers to allocate their entire endowment to a single good, depending on their valuation of environmental quality relative to cost. This leads to three distinct consumer types:

- Green Consumers: They only consume green goods ($g_i^* > 0$, $b_i^* = 0$).
- Brown Consumers: They only consume brown goods ($g_i^* = 0$, $b_i^* > 0$).
- Mixed Consumers: They consume both goods ($g_i^* > 0$, $b_i^* > 0$).

To determine which category each consumer falls into, it is useful to compare their Marginal Rate of Substitution (MRS) between environmental quality and private consumption to the relative price of green versus brown goods:

$$\frac{U_Y}{U_{x_i}} \gtreqless \frac{P_g - P_b}{P_g}$$

If the left-hand side of the previous expression is greater than the right-hand side, the consumer is willing to pay more than the price difference between g_i and b_i to preserve one unit of Y. In this case, the consumer will exclusively purchase green goods and is classified as a Green

Consumer. Conversely, if the left-hand side is less than the right-hand side, the consumer values environmental quality less than the additional cost of green goods and will only purchase brown goods, thereby being classified as a Brown Consumer. If both sides of the expression are equal, the consumer's willingness to pay to preserve one unit of Y exactly matches the price difference between the two goods. In this situation, the consumer chooses a combination of green and brown goods to maximize utility and is classified as a Mixed Consumer.

While the previous analysis focuses on individual choices and the resulting heterogeneity in consumption patterns, it is also important to consider how a social planner might evaluate and regulate these outcomes from a welfare-maximizing perspective. Suppose that there is a regulatory agency whose goal is to maximize social welfare. To that extent, the agency seeks to determine the optimal level of environmental quality given the total income in the economy ($\sum_{i=1}^{n} w_i$) and the market prices of the green (g_i) and brown goods (b_i). The overall social budget is given by: $\sum_{i=1}^{n} w_i = P_g \sum_{i=1}^{n} g_i + P_b \sum_{i=1}^{n} b_i$.

Using the identities from the baseline model —namely $x_i = g_i + b_i$ and $Y = Y^0 - \sum_{i=1}^{n} b_i$— it is possible to express the social budget in terms of total private consumption and environmental quality.

$$M = P_x \sum_{i=1}^{n} x_i + P_Y Y$$

where $M = \sum_{i=1}^{n} w_i + P_Y Y^0$, $P_x = P_g$, and $P_Y = P_g - P_b$. The regulatory agency evaluates a social welfare function $S(\cdot)$, which includes the utility of every consumer as a separate argument. Since utility functions are heterogeneous across individuals, the social welfare function reflects this diversity by incorporating each utility in an independent manner. The function $S(\cdot)$ shares the same properties as the individual utility functions: it is quasiconcave, continuous, and strictly increasing in each of its arguments. The objective of the regulatory agency is to select the level of total private consumption for each consumer and the aggregate level of environmental quality Y, in order to maximize social welfare subject to the overall budget constraint:

$$\max_{x_1, x_2, \ldots, x_n, Y} \left\{ S\big(U_1(x_1, Y), U_2(x_2, Y), \ldots, U_n(x_n, Y)\big) \middle| M = P_x \sum_{i=1}^{n} x_i + P_Y Y \right\}$$

Assuming that the optimal level of Y is located somewhere in the interior,[2] then the socially optimal allocation is the unique solution to the following $n+1$ first-order conditions:

$$\frac{\partial S}{\partial U_i} \cdot \frac{\partial U_i}{\partial x_i} - \lambda P_x = 0, \qquad i = 1,2,\dots,n$$
$$\sum_{i=1}^{n} \frac{\partial S}{\partial U_i} \cdot \frac{\partial U_i}{\partial Y} - \lambda P_Y = 0$$

where λ is the Lagrange multiplier associated with the social budget constraint. From the previous set of equations, the optimal marginal rate of substitution for each consumer i must be equal to:

$$\frac{\partial U_i / \partial x_i}{\partial U_i / \partial Y} = \frac{P_x}{P_Y}\left[1 + \frac{S_{Y_{-i}}}{\frac{\partial S}{\partial U_i} \cdot \frac{\partial U_i}{\partial Y}}\right], \qquad i = 1,2,\dots,n \qquad (3.1)$$

where $S_{Y_{-i}} = \sum_{j \neq i} \frac{\partial S}{\partial U_j} \cdot \frac{\partial U_j}{\partial x_j} > 0$. The result in (3.1) highlights two key differences between the social optimum and the competitive equilibrium. First, achieving the social optimum requires all consumers to have the same marginal rate of substitution between private consumption and environmental quality. This condition holds for Mixed consumers but not for Green or Brown consumers, whose marginal rates of substitution deviate from the relative price ratioP_x/P_Y. Second, because brown goods generate a negative externality, the socially optimal outcome requires a marginal rate of substitution that exceeds this price ratio. Therefore, a policy aimed at achieving the social optimum must employ two distinct instruments, one to equalize marginal rates of substitution across consumers and another to internalize the environmental externality.

To address the externality associated with brown goods consumption, I propose implementing a permit market based on the individual transferable quotas (ITQs) model. Under this system, a regulatory agency allocates a fixed number of permits to consumers, thereby limiting the total quantity of brown goods consumed. Once consumers exhaust their initial allocation, they are not allowed to purchase additional units of the brown good unless they acquire permits from others who have unused allocations. This trading mechanism enables consumers who consume fewer

[2] The maximum value that Y can have is Y^0, and it occurs if there is no consumption of b_i in the economy. The minimum value would be $Y^0 - \frac{\sum_{i=1}^{n} w_i}{P_b}$ and it occurs if there is no consumption of g_i in the economy. Therefore, the previous statement refers to the scenario where the regulatory agency will look to have a positive consumption of both g_i and b_i.

brown goods to earn additional income by selling their surplus permits to those who demand more. The revenue generated from these sales can then be used to increase their consumption of either green or brown goods.

The cost of acquiring a permit effectively increases the price of the brown good. The permit price is determined by market forces, and all consumers make their consumption decisions based on this publicly observed price. The proceeds from permit sales go directly to the sellers, removing the need for the regulatory agency to redistribute revenues. The permit market thus offers two instruments: the total quantity of permits, which controls the externality, and the market-determined permit price, which enables redistribution through voluntary exchange. By capping the aggregate consumption of brown goods and allowing decentralized transfers through trading, this policy can improve both environmental quality and the overall allocation of resources.

3.2 The market for permits

Building on the baseline model, the regulatory agency seeks to guarantee a minimum level of environmental quality in the economy by setting a cap on the total consumption of brown goods. To enforce this cap, the agency distributes an initial endowment of permits to each consumer i. Each permit grants the right to purchase one unit of b_i at the market price P_b. Therefore, the maximum number of units of the brown good that consumers can buy is equal to the number of permits initially received. Let $\bar{q}_i$ denote the initial permits received by consumer i and define the vector of all permit allocations by the regulator as $\bar{\boldsymbol{Q}} = (\bar{q}_1, \bar{q}_2, \dots, \bar{q}_n)$.

Consumers who wish to consume more brown goods than initially permitted may acquire additional permits from others. The regulatory agency allows consumers to negotiate the permit price P_q freely in a decentralized market. I assume that all individuals behave as price takers, that information about trades is publicly available, transaction costs are negligible, and that P_q adjusts to its competitive equilibrium level. Revenue earned from permit sales can be used by sellers to increase their total private consumption. In the following sections, I analyze the decision problems faced by buyers and sellers separately.

Before doing so, I introduce an additional assumption regarding individual preferences over environmental quality. Specifically, I assume that the individual demand for environmental quality satisfies the following property:

$$\frac{\partial Y}{\partial Y_{-i}} \in (0,1) \quad (3.2)$$

where $Y_{-i} = Y^0 - \sum_{j \neq i} b_j$ represents the level of environmental quality that consumer i takes as given when making individual consumption decisions. In other words, Y_{-i} captures the environmental outcome determined by the behavior of all other consumers and is treated as exogenous from the perspective of consumer i. The property in (3.2) is referred to as the normality assumption, as it implies that both x_i and Y are strictly normal goods.

3.3. The Buyers' Problem

Consider a scenario where the regulator allocates an initial endowment of $\bar{q}_i$ permits to consumer i $(\bar{q}_i \geq 0)$. Suppose that consumer i desires to consume a quantity of b_i that exceeds this initial allocation. In such a situation, the consumer has an incentive to buy additional permits in the market. Let q_i^d denote the quantity of permits demanded by consumer i in addition to the initial endowment. Purchasing additional permits requires allocating part of the individual income to permit expenditures. However, doing so allows the consumer to increase brown good consumption beyond the level permitted by $\bar{q}_i$, up to a maximum of $b_i \leq \bar{q}_i + q_i^d$. Given this context, the consumer faces a utility maximization problem that involves a trade-off between acquiring additional permits and spending on goods, subject to the budget and brown consumption constraints.

$$\max_{x_i, Y, g_i, b_i, q_i^d} \left\{ U_i(x_i, Y) \,\middle|\, \begin{array}{ll} w_i = P_g g_i + P_b b_i + P_q q_i^d, & b_i \leq \bar{q}_i + q_i^d, \\ x_i = g_i + b_i, & Y = Y_{-i} - b_i - \dfrac{P_q q_i^d}{P_b}, \\ & g_i, b_i, q_i^d \geq 0 \end{array} \right\} \quad (3.3)$$

This utility-maximization problem differs from the baseline model in three aspects. First, the budget constraint now considers the possibility that consumers allocate a portion of their income to the purchase of permits (q_i^d), to increase their consumption of brown goods. Second, consumption of the brown good is explicitly constrained by the total number of permits available to consumer i, whether through initial allocation or market purchases. Third, buyers must now consider the negative externality associated with their actions: the revenue from permit sales is used by sellers to increase their own consumption of brown goods, which in turn reduces overall environmental quality.[3]

Within this framework, consumers must decide whether it is optimal to acquire additional permits and, if so, how many to purchase. From (3.3), the term q_i^d appears only in the budget constraint and the brown consumption constraints. Therefore. the individual demand for permits can be derived by the corresponding first-order condition:

$$\lambda_w P_q = \lambda_b + \mu_q$$

where λ_w and λ_b are the Lagrange multipliers associated to each of the constraints and μ_q is the Kuhn-Tucker multiplier related with the non-negativity constraint on q_i^d. Note that if the constraint $b_i \leq \bar{q}_i + q_i^d$ is not binding, then $\lambda_b = 0$. In this case, the first-order condition with respect to q_i^d requires $\mu_q > 0$ and $q_i^{d^*} = 0$.[4] This situation implies that consumers can satisfy their demand for brown goods with the initial allocation, and their behavior aligns with that of the baseline model, where permit trading does not take place.

In contrast, if $q_i^{d^*} > 0$, it follows that $b_i = \bar{q}_i + q_i^d$. This relationship implies that the individual demand for permits must fully offset the demand for the brown good, considering the initial allocation $\bar{q}_i$. By substituting this equality into the budget constraint, we can derive the following condition:

$$w_i - P_b \bar{q}_i = P_g g_i + (P_b + P_q) q_i^d$$

When consumers act as buyers in the permit market, they first use their initial allocation to acquire as many units of b_i as possible at the prevailing priceP_b. After exhausting this initial allocation, any remaining income can be used for either consuming green goods or to acquire additional permits, which would allow for further consumption of brown goods. Since obtaining one additional unit of the brown good requires one permit, the effective cost of consuming an extra unit is the sum $P_b + P_q$, combining the market price of the brown good and the permit price.

By expressing the income constraint in this form, it is possible to employ the identities stated in (3.3) to reformulate the utility-maximizing problem as follows:

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[3] In the following section, I provide an explanation for why sellers allocate the revenue from selling permits towards increasing their consumption of brown goods.

[4] The previous condition will not hold in the special case where $P_q = 0$. In this situation, there would be free disposal of permits and consumers would get any amount of them without impacting their optimal choice of g_i^* and b_i^*.

$$\max_{x_i,Y} \left\{ \left. U_i(x_i,Y) \right| \; m_i^d = P_x^d x_i + P_Y^d Y, \quad Y_{-i} - x_i \le Y \le Y_{-i} - \bar{q}_i \quad \right\} \tag{3.4}$$

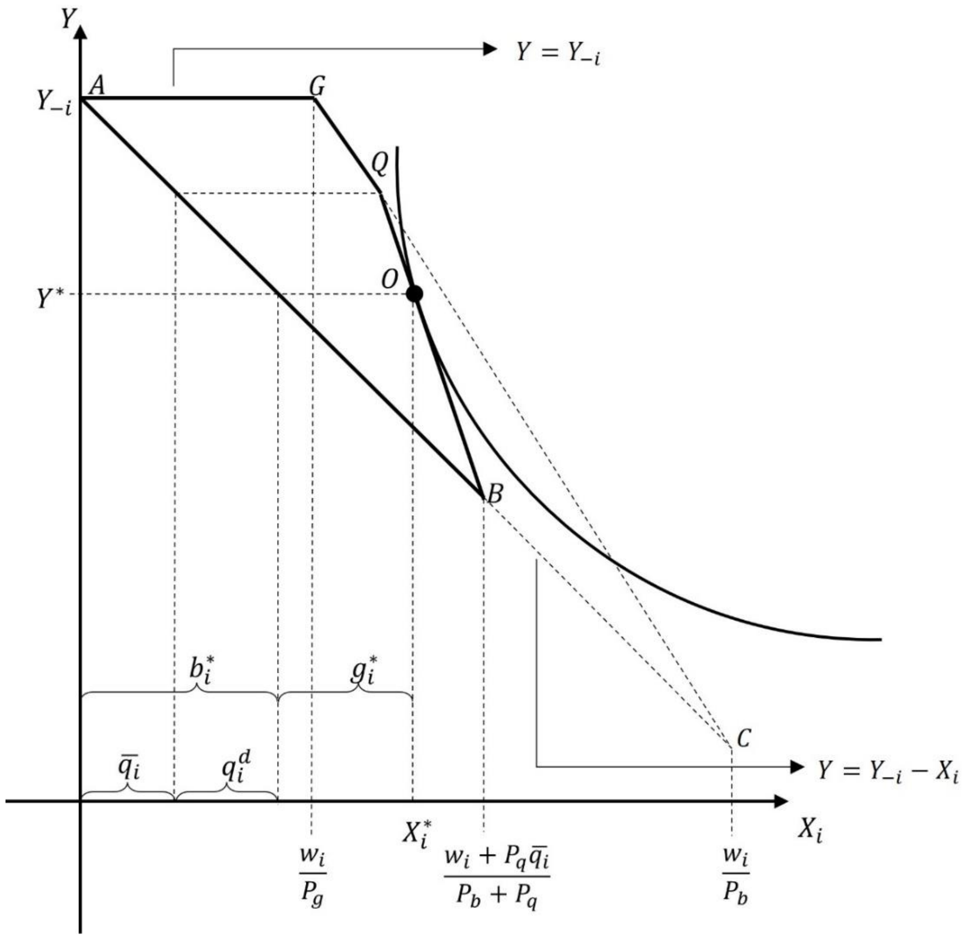
where $m_i^d = w_i + \frac{P_g P_q}{P_b + P_q} \bar{q}_i + P_Y^d Y_{-i}$, $P_x^d = P_g$, $P_Y^d = \frac{P_b}{P_b + P_q}\left(P_g - P_b - P_q\right)$, and the superscript d refers to the scenario where consumer i has a positive demand for permits. As such, it is possible to express the problem only in terms of the full-income (m_i^d) and the implicit prices (P_x^d and P_Y^d). The full-income includes not only the consumer's monetary endowment but also the value of the initial allocation of permits, which can be used directly for consumption of brown goods or sold in the market. This augmented budget reflects the total economic resources available for consumption decisions. The implicit price of environmental quality accounts for the cost of acquiring permits to consume brown goods. The inequality $Y \le Y_{-i} - \bar{q}_i$ comes from the non-negativity constraint on q_i^d and captures the fact that consumer i will use all the initial permits to consume b_i . The inequality $Y \ge Y_{-i} - x_i$ comes from the non-negativity constraint on g_i and captures the situation where all the income is used to consume b_i (via the acquisition on permits). Alternatively, this inequality also states that environmental quality cannot be lower than the impact caused by the private consumption of consumer i.

Figure 3.1 illustrates the budget set and the optimal consumption bundle in terms of (x_i, Y). The feasible set is convex and defined by the trapezoid AGQB. The segment AC has a slope of -1 and captures the impact of the consumption of the brown good on environmental quality. The segment GQB represents the budget constraint and is composed of two linear parts, split by the kink point Q. The segment GQ reflects the case when consumer i satisfies the consumption of the brown good with the initial allocation of permits. Beyong point Q, the segment QB corresponds to the situation in which the consumer must buy a permit for every additional unit of the brown good. As a result, the marginal cost of private consumption increases beyond Q due to the permit price. The triangle BQC represents the area that would have been feasible in the absence of the permit constraint. This triangle highlights the distortion to individual decisions introduced by the permit system.

If the optimal consumption bundle from solving (3.4) lies along the segment QB, then the consumer is actively participating in the permit market and purchasing additional permits. Under this condition, the demand for environmental quality can be expressed as a function of full income and the vector of implicit prices $\mathbf{P^d} = \left(P_x^d, P_Y^d\right)$:

$$Y = y_i\left(m_i^d, \mathbf{P^d}\right) \tag{3.5}$$

Figure 3.1 Budget Set and Consumption Bundle (Permits Buyer)



Expression (3.5) is useful to recover the demand for permits by using the identity $Y_{-i} = Y + \bar{q}_i + q_i^d$.[5] By assumption, environmental quality is a normal good, which implies that (3.5) is an increasing function with respect to m_i^d. As such, there exists an inverse function ${y_i^d}^{-1}(Y, \mathbf{P^d})$ with respect to the full income that is equal to:

$$
\begin{aligned}
{y_i^d}^{-1}(Y, \mathbf{P^d}) &= m_i^d \\
&= w_i + P_q\bar{q}_i + P_Y^d Y_{-i}
\end{aligned}
$$

[5] This identity comes from rearranging $Y = Y_{-i} - b_i$ and by assuming that the constraint $b_i \leq \bar{q}_i + q_i^d$ holds with equality.

The interpretation of ${y_i^d}^{-1}\left(Y, \mathbf{P^d}\right)$ is that it gives the full income required to sustain a demand of Y units of environmental quality when facing the price vector $\mathbf{P^d}$. From this previous expression, it is possible to obtain an identity that sets q_i^d as function that depends on the other parameters of the model.

$$q_i^d = \frac{y_i^{-1}\left(Y, \mathbf{P^d}\right) - w_i}{P_Y^d} - Y - \left(\frac{P_g - P_b}{P_Y^d}\right)\bar{q}_i \qquad (3.6)$$

The right-hand side of expression (3.6) includes Y as one of its arguments. Thus, it is possible to interpret q_i^d as a value that is conditional on a given level of Y.[6] Since the optimal value of Y is restricted to a compact set by (3.4), the range of expression (3.6) must also be bounded. Let the function $q_i^d(Y, \bar{q}_i)$ represent the conditional demand for permits of consumer i, suppressing notation for income and prices. The following proposition summarizes the main properties of $q_i^d(Y, \bar{q}_i)$:

Proposition 3.1: *If environmental quality is a normal good and* q_i^d *has a representation as in* (3.6)*, then consumer i has a conditional demand for permits* $q_i^d(Y, \bar{q}_i)$ *with the following properties:*

- *i)* $q_i^d(Y, \bar{q}_i)$ *is a weakly increasing function in Y.*
- *ii)* $q_i^d(Y, \bar{q}_i)$ *is a weakly decreasing function in* $\bar{q}_i$*.*
- *iii) For all* $w_i > 0$*, there exists a value* $P_{q_i}^{d-max} < P_g - P_b$ *such that* $q_i^d(Y, \bar{q}_i) = 0$ *if* $P_q \geq P_{q_i}^{d-max}$.
- *iv) There exists an income level* $w_i^{CB}(Y, \bar{q}_i)$ *such that* $q_i^d(Y, \bar{q}_i) = \frac{w_i - P_b\bar{q}_i}{P_b + P_q}$ *if* $w_i \leq w_i^{CB}(Y, \bar{q}_i)$

The result in part i) follows from assumption (3.2) that guarantees that both x_i and Y are normal goods. An exogenous increase in a given level of Y implies that the full-income also increases. Since income, prices, and the initial allocation of permits are fixed, the only channel through which full income can increase is a higher level of environmental quality provided by

[6] Cornes and Hartley (2007) give expression (3.6) the interpretation of a replacement function. The reasoning for this name is that if there exists an exogenous change that increases Y in $\hat{Y}$ units, then the demand of consumer i would change in the same amount to keep the level of Y fixed at its target value.

others (i.e., an increase in Y_{-i}). In response, the consumer will demand more brown goods, though not enough to fully offset the environmental improvement. As a result, the individual demand for permits increases, making q_i^d a weakly increasing function of Y.

Part ii) reflects the idea that permit demand is inversely related to the initial allocation $\bar{q}_i$. This is consistent with the structure of equation (3.6). consumers who receive more permits at the outset have less incentive to purchase additional permits, as their consumption needs for brown goods are partially met by the allocation. Thus, permit demand is weakly decreasing in $\bar{q}$. Part iii) establishes a price threshold above which the consumer will no longer find it optimal to buy permits. When the permit price approaches or exceeds the price difference between green and brown goods, the cost of acquiring additional permits becomes prohibitively high. In this case, the consumer reallocates all remaining income to green goods, and demand for permits falls to zero. This condition confirms that the permit market only functions within a specific price range, below $P_g - P_b$.

Part iv) captures the behavior of income-constrained consumers. When income is sufficiently low, the consumer may choose to acquire as many permits as possible, even if doing so means foregoing all consumption of green goods. In this scenario, the demand for permits becomes linear in income, and is given by $q_i^d(Y, \bar{q}_i) = \frac{w_i - P_b\bar{q}_i}{P_b + P_q}$, which reflects the need of this constrained consumer to prioritize brown good consumption due to its lower price.

To distinguish between consumers in terms of their demand for permits, I will use two categories: regular buyers (RB) and constrained buyers (CB). Regular buyers are those whose demand for permits can be modeled by equation (3.6). On the other hand, while constrained buyers are those who have a demand constrained by their low income and can be represented by the equation $q_i^d(Y, \bar{q}_i) = \frac{w_i - P_b\bar{q}_i}{P_b + P_q}$.

3.4 The Sellers' Problem

From the perspective of a seller, receiving an initial allocation of permits $\bar{q}_i$ is valuable not only as a means to enable personal consumption of brown goods, but also because these permits have trade value that can be used to increase the individual budget. Consumers who choose to sell a portion of their permits receive a payment equal to P_q for each unit sold. However, sellers must carefully evaluate the trade-off between generating income through permit sales and satisfying their own consumption needs. They should avoid selling more permits than they intend to use themselves, as doing so would constrain their own consumption of the brown good. Moreover, sellers must recognize that any permit sold is likely to be used by buyers to increase brown good consumption, which contributes to environmental degradation. Therefore, sellers must consider not only the monetary gain from trading permits but also the environmental cost associated with their use.

Taking these factors into account, the utility maximization problem faced by a seller can be formally expressed as follows:

$$\max_{x_i,Y,g_i,b_i,q_i^s}\left\{U_i(x_i,Y)\middle|\begin{array}{ll}w_i+P_q q_i^s=P_g g_i+P_b b_i, & \bar{q}_i\geq b_i+q_i^s,\\ x_i=g_i+b_i, & Y=Y_{-i}-b_i-q_i^s,\\ & g_i,b_i,q_i^s\geq 0\end{array}\right\} \quad (3.7)$$

where q_i^s denotes the number of permits that consumer i offers for sale in the market. If the optimal choice satisfies that $q_i^{s^*}=0$ is optimal, then the solution to (3.7) coincides with that of the baseline model, as no trading occurs. In the more relevant case where $q_i^{s^*}>0$, sellers must decide how to allocate total income, including revenue from sales, between g_i and b_i. Assuming an interior solution, where the initial endowment of permits exceeds the sum of permits sold and brown goods consumed (i.e., $\bar{q}_i>b_i+q_i^s$), the individual demands for g_i and b_i, along with the supply of q_i^s can be characterized by equating the marginal utility per unit of money spent on each good:[7]

$$\frac{U_x+\mu_g}{P_g}=\frac{U_x-U_Y+\mu_b}{P_b}=\frac{U_Y}{P_q}$$

Note that the previous expressions include the Kuhn–Tucker multipliers μ_g and μ_b to account for the possibility of having an optimal consumption of $g_i^*=0$ or $b_i^*=0$. Solving for U_x gives a unique relationship involving the marginal utility of environmental quality and the three market prices P_g, P_b, and P_q:

$$\frac{P_g}{P_q}U_Y-\mu_g=\frac{P_b+P_q}{P_q}U_y-\mu_b$$

[7] To simplify the maximization problem, it possible to substitute the identities $x_i=g_i+b_i$ and $Y=Y_{-i}-b_i-q_i^s$ into the utility function and use the following Lagrangian function:

$$\mathrm{L}=U_i(g_i+b_i,Y_{-i}-b_i-q_i^s)+\lambda_w\left[w_i+P_q q_i^s-P_g g_i-P_b b_i\right]+\mu_g g_i+\mu_b b_i$$

This condition is useful for understanding the incentives that sellers face regarding the consumption of green and brown goods, given the market price for permits. If the seller chooses to consume the green good, then $\mu_g = 0$.[8] However, such a consumption choice is optimal only if the permit price satisfies $P_q \geq P_g - P_b$. As shown in the analysis of the buyer's problem, if the permit price exceeds than the price difference between green and brown goods, there will be no positive demand for permits. For permit trading to occur, the price of permits must satisfy $P_q < P_g - P_b$. This condition implies that it is never optimal for a seller to consume the green good. As a result, sellers satisfy their private consumption entirely through brown goods, so that $x_i^* = b_i^*$. Under this condition, the budget constraint simplifies to $w_i + P_q q_i^s = P_b b_i^*$. With the income constraint expressed in this way, and using the identities in (3.7), the utility maximization problem for seller simplifies to:

$$\max_{x_i, Y} \{ \; U_i(x_i, Y) | \; m_i^s = P_x^s x_i + P_Y^s Y, \quad Y_{-i} - \bar{q}_i \leq Y \leq \; Y_{-i} - x_i \; \} \tag{3.8}$$

where $m_i^s = w_i + P_Y^s Y_{-i}$, $P_x^s = P_b + P_q$, $P_Y^s = P_q$. The superscript s refers to the to the case in which consumer i offers a positive number of permits in the market. As in the buyer's problem, it is possible to express the seller's problem in terms of the full-income (m_i^s) and the implicit prices (P_x^s and P_Y^s). In this scenario, the full-income includes both personal income and the monetary value of the environmental quality exogenously provided by others.

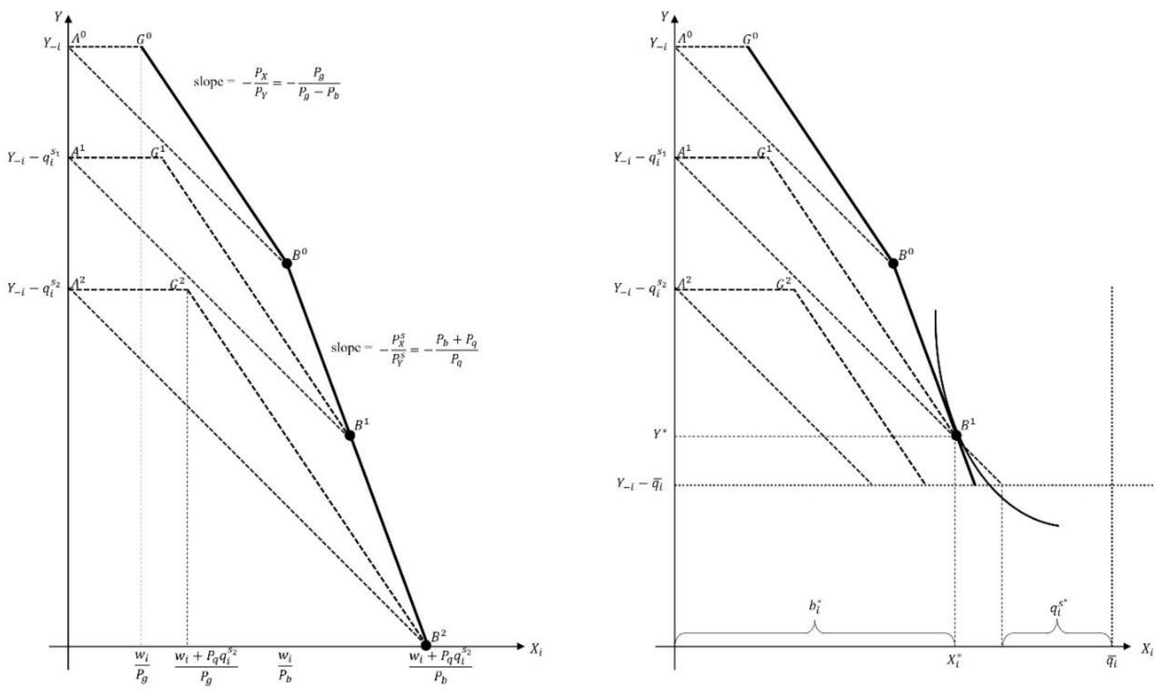
However, the implicit prices for both X_i and Y differ from the earlier case. The implicit price of x_i is $P_b + P_q$, reflecting both the purchase price of the brown good and the opportunity cost of using a permit rather than selling it. The implicit price of Y is P_q, since that is the amount the consumer is effectively willing to accept in exchange for a reduction of one unit of environmental quality. The constraint $\bar{q}_i \geq b_i + q_i^s$ implies the inequality $Y_{-i} - \bar{q}_i \leq Y$, which defines the minimum level of environmental quality that can be achieved when consumer i fully uses all the individual permits, whether through consumption or sale. Conversely, the inequality $Y \leq \; Y_{-i} - x_i$ follows from the non-negativity constraint on q_i^s. It indicates that environmental quality must decline by at least the amount of private consumption undertaken by consumer i, and

[8] In this case, the consumption of the brown good is either positive of zero and $\mu_b \geq 0$. Therefore, the previous expression simplifies to:

$$0 \leq \mu_b = \frac{P_q - (P_g - P_b)}{P_q} U_Y$$

this condition binds when the consumer uses all permits for personal consumption and refrains from trading.

Figure 3.2: Budget Set and Consumption Bundle (Permits Seller)



a) Budget set without considering the initial allocation of permits.

b) Optimal consumption bundle considering the initial allocation of permits

Panel a) of Figure 3.2 illustrates how the budget set changes as consumer i sells permits. Initially, the consumer can choose any bundle within the triangle $A^0G^0B^0$. When consumer i sells $q_i^{s_1}$ permits, the budget set shifts downward to the larger triangle $A^1G^1B^1$. In this case, point G^1 lies to the right of G^0, indicating that the consumer receives income from selling the permits. However, it lies below the line G^0B^0, reflecting that P_q is less than the price difference $P_g - P_b$. If the consumer sells $q_i^{s_2}$ permits instead, the budget set expands further down to triangle $A^2G^2B^2$.

The consumption bundles in points B^0, B^1, and B^2 in the (x_i, Y) space represent outcomes where the consumer spends all available income on brown goods, subject to the number of permits sold. Therefore, the line connecting these points represents the budget line, with a slope equal to $-P_x^S/P_Y^S$. Notably, point B^0 corresponds to the case in which the consumer uses all income on brown goods without selling any permits, leading to an environmental quality level of $Y = Y_{-i} - x_i$.

Panel b) depicts how the initial allocation $\bar{q}_i$ constraints the budget set and influences the optimal consumption bundle. The horizontal line at $Y = Y_{-i} - \bar{q}_i$ serves as an upper bound, excluding any consumption bundles below it as infeasible. Suppose the optimal consumption is represented by point B^1. The optimal number of permits to sell corresponds to the horizontal distance between the intersection of line A^1B^1 with the constraint line at $Y = Y_i - \bar{q}_i$ and the vertical line at $x_i = \bar{q}_i$. This distance reflects the permits that consumer i must sell to be able to afford the optimal bundle. As a result, the consumer retains a positive number of permits that are neither sold nor used for brown goods consumption.

Assuming that point B^1 is the solution to (3.8), then it is possible to express the demand for environmental quality as a function of the full income and the vector of implicit prices $\mathbf{P}^s = (P_x^s, P_Y^s)$:

$$Y = y_i^s(m_i^s, \mathbf{P}^s) \tag{3.9}$$

By assumption, environmental quality is a normal good, meaning that (3.9) is an increasing function with respect to m_i^s. As such, there exists an inverse function $y_i^{d^{-1}}(Y, \mathbf{P}^\mathbf{d})$ with respect to the full income that is equal to:

$$\begin{aligned} y_i^{s^{-1}}(Y, \mathbf{P}^\mathbf{s}) &= m_i^s \\ &= w_i + P_Y^s Y_{-i} \end{aligned}$$

Similar to the buyer's problem, the expression $y_i^{s^{-1}}(Y, \mathbf{P}^\mathbf{s})$ represents the level of full income required for consumer i to sustain a given demand of Y units of environmental quality, given the price vector $\mathbf{P}^\mathbf{s}$. From the identity $Y = Y_{-i} - b_i - q_i^s$ and the budget constraint $w_i + P_q q_i^s = P_b b_i$, it is possible to express the supply of permits in terms of $y_i^{s^{-1}}(Y, \mathbf{P}^\mathbf{s})$. This yields a closed-form expression for the quantity of permits that consumer i is willing to sell, conditional on the desired level of environmental quality and the prevailing implicit prices:

$$q_i^s = \frac{P_x^s - P_Y^s}{P_x^s}\left[\frac{y_i^{s^{-1}}(Y, \mathbf{P}^\mathbf{s})}{P_Y^s} - Y\right] - \frac{w_i}{P_Y^s} \tag{3.10}$$

Since expression (3.10) includes Y as one of its arguments, it is possible to interpret q_i^s as the conditional supply of permits given a specific level of Y. Because the feasible levels of Y are restricted to a compact set by (3.8), the range of expression (3.10) must also be bounded.[9] Let

[9] This situation is equivalent to what occurs in the buyer's problem.

$q_i^s(Y, \bar{q}_i)$ denote the conditional supply of permits for consumer i, suppressing notation for income and prices. The following proposition summarizes the key properties of $q_i^s(Y, \bar{q}_i)$:

Proposition 3.2: *If environmental quality is a normal good and* q_i^s *has a representation as in* (3.10)*, then consumer i has a conditional supply for permits* $q_i^d(Y, \bar{q}_i)$ *with the following properties:*

1. *For all* $w_i > 0$*, there exists a value* $P_{q_i}^{s-min} > 0$ *such that* $q_i^s(Y, \bar{q}_i) = 0$ *if* $P_q \leq P_{q_i}^{s-min}$.
2. *There exists a maximum number of permits that consumer* i *is willing to use, whether it is for consumption or for trading purposes. If consumer* i *receives fewer initial permits than this amount, then an income level* $w_i^{CS}(Y, \bar{q}_i)$ *exists such that* $q_i^s(Y, \bar{q}_i) = \frac{P_q\bar{q}_i - w_i}{P_b + P_q}$ *if* $w_i \leq w_i^{CS}(Y, \bar{q}_i)$
3. $q_i^s(Y, \bar{q}_i)$ *is a weakly increasing function in Y.*

The first part of the proposition establishes that for any positive income level $w_i > 0$, there exists a minimum permit price such that consumers will choose not to sell any permits. At such low prices, the revenue from selling permits is insufficient to justify giving up consumption of brown goods or accepting the associated environmental cost. As a result, the consumer retains all permits for personal use. This condition defines the lower bound at which selling becomes worthwhile.

The second part of the proposition addresses cases in which the consumer receives fewer permits than they would ideally like to use, either for consumption or sale, and is also constrained by a low income level. In this context, the consumer may be compelled to sell a portion of their permits to afford basic private consumption. When income falls below a threshold $w_i^{CS}(Y, \bar{q}_i)$, the conditional supply of permits follows a linear expression $q_i^s(Y, \bar{q}_i) = \frac{w_i + P_q\bar{q}_i}{P_b}$. This expression reflects the trade-off faced by the seller: as income decreases, the individual must sell more permits to maintain minimum consumption. Here, permit sales serve as a compensatory mechanism for income constraints

Finally, the proposition asserts that the conditional supply function $q_i^s(Y, \bar{q}_i)$ is weakly increasing in Y. This property implies that when environmental quality improves, due to lower consumption of brown goods by others, the marginal utility of further environmental preservation diminishes. As a result, the consumer becomes more willing to convert environmental quality into additional income or private consumption by supplying more permits to the market.

To distinguish between sellers in terms of their supply for permits, I will use two categories: regular sellers (RS) and constrained sellers (CS). Regular sellers are those whose supply for permits can be modeled by equation (3.10). On the other hand, constrained sellers are those who have a supply constrained by their low income and can be represented by the equation $q_i^d(Y, \bar{q}_i) = \frac{w_i - P_b \bar{q}_i}{P_b + P_q}$.

3.5 Market Equilibrium

In the previous sections, I identified the conditions under which consumers have incentives to trade permits among each other. A functioning market requires the presence of consumers who wish to increase their consumption of the brown good but are limited by an insufficient initial allocation of permits. At the same time, there must be consumers who receive more permits than they can afford to use, thereby generating a supply of permits. To achieve an efficient outcome through this market, the regulatory agency must design the initial allocation in such a way that low-income consumers are able to participate as sellers and benefit from the resulting income transfers.

If the regulatory agency assigns permits in a manner that induces at least one consumer to act as a buyer and another as a seller, then it is possible to demonstrate the existence of a market equilibrium with tradable permits.

***Definition 3.3**: A market equilibrium with tradable permits is a price P_q^*, an allocation $\big((g_1^*, b_1^*), (g_2^*, b_2^*), \dots, (g_n^*, b_n^*), Y^*\big)$, and an array of trades $\bar{\boldsymbol{Q}} = (q_1^*, q_2^*, \dots, q_n^*)$ such that:*

1. *for all* $i = 1,2, \dots, n$: $U_i(g_i^* + b_i^*, Y^*) \geq U_i(g_i + b_i, Y)\ \forall (g_i + b_i, Y) \in \mathbb{R}_+ \times \mathbb{R}$ *for which* $w_i = P_g g_i + P_b b_i + P_q^* q_i^*$
2. $Y^0 - \sum_{i=1}^n b_i^* = Y^*$
3. $Q^D(P_q^*, Y^*) = Q^S(P_q^*, Y^*) > 0$

where $Q^D(P_q^*, Y^*) = \sum_{i \in D} q_i^d(Y, \bar{q}_i)$ and $Q^S(P_q^*, Y^*) = \sum_{j \in S} q_j^s(Y, \bar{q}_j)$ denote the aggregate demand and aggregate supply of permits. The term q_i^* represents the net number of permits that consumer i trades in the market and is defined as $q_i^* = q_i^{d^*} - q_i^{s^*}$. This value is positive if consumer i buys permits, negative if consumer i sells permits, and zero if consumer i does not participate in the market.

The first condition of Definition 3.3 ensures that each consumer maximizes utility and has no incentive to deviate from their chosen allocation. The second condition implies that the aggregate consumption of the brown good results in a Nash Equilibrium level of environmental quality. The following two lemmas are useful to help establish that, for any given permit price P_q, there exists a Nash Equilibrium in the permit market.

Lemma 3.4: *$b_i(Y, \bar{q}_i)$ is a continuous and weakly increasing function in Y $\forall\ i = 1, \dots, n$.*

Lemma 3.5*: Let $P_q \in (0, P_g - P_b)$, then there exists a unique value Y^* that satisfies the following equation:*

$$Y^0 - \sum_{i=1}^{n} b_i(Y^*, \bar{q}_i) = Y^* \qquad (3.11)$$

The ***first*** condition of Definition 3.3 ensures that each consumer maximizes utility and has no incentive to deviate from their chosen allocation. The second condition implies that the aggregate consumption of the brown good results in a Nash Equilibrium level of environmental quality. The following two lemmas are useful to help establish that, for any given permit price P_q, there exists a Nash Equilibrium in the permit market.

Lemma 3.4 holds true even for non-participants in the market, as they follow the behavior outlined in the baseline model. For buyers of permits, the identity $b_i = \bar{q}_i + q_i^d$ establishes that $b_i(Y, \bar{q}_i)$ shares all the properties from the function $q_i^d(Y, \bar{q}_i)$, which are described in Proposition 3.1. A similar situation occurs with sellers, as the identity $P_b b_i = w_i + P_q q_i^s$ guarantees that the function $b_i(Y, \bar{q}_i)$ shares all the properties from the function $q_i^s(Y, \bar{q}_i)$, which was described in Proposition 3.2. Therefore, since $b_i(Y, \bar{q}_i)$ is an increasing function in Y, then it is useful to establish the result from Lemma 3.5.

The third condition of Definition 3.3, which characterizes a market equilibrium with tradable permits, requires that aggregate demand for permits equals aggregate supply. Let $P_q^{d-max}(Y^*)$ denote the lowest price across all potential buyers at which the aggregate demand for permits is zero, given an environmental quality level Y^*. Similarly, let $P_q^{s-min}(Y^*)$ denote the highest price across all potential sellers at which the aggregate supply of permits is zero at the same level Y^*. The following Lemma establishes that there exists a permit price $P_q \in [P_q^{s-min}(Y^*), P_q^{d-max}(Y^*)]$ such that the market clears.

Lemma 3.6: *Let* Y^* *be defined as in* (3.11) *and* $P_q^{d-max}(Y^*) > P_q^{s-min}(Y^*)$ *so that trading occurs in the market. Then there exists a market-clearing price* P_q^* *such that:*

$$Q^D\left(P_q^*, Y^*\right) = Q^S(P_q^*, Y^*)$$

Therefore, I can draw on the the results from Lemma 3.5 and Lemma 3.6 to prove that, provided that consumers have incentives to trade permits, a market equilibrium with tradable permits exists:

Proposition 3.7: *If* $P_q^{d-max}(Y^*) > P_q^{s-min}(Y^*)$ *, then* a *market equilibrium with tradable permits exists*.

It is important to note that Proposition 3.7 guarantees only the existence of an equilibrium. It does not imply that any initial permit allocation selected by the regulatory agency will necessarily result in a socially efficient outcome. Thus, the regulator must carefully evaluate different allocation strategies to identify the one that maximizes social welfare. In the following analysis, I introduce specific market assumptions to examine how variations in the initial distribution of permits affect the resulting equilibrium and overall efficiency:

A.2.1. *Law of Demand for permits:* There is an inverse relationship between the price and the quantity demanded of permits. $\left(\frac{\partial Q^D(P_q,Y)}{\partial P_q} < 0\right)$

A.2.2. *Law of Supply for permits:* There is a direct relationship between the price and the quantity supplied of permits. $\left(\frac{\partial Q^S(P_q,Y)}{\partial P_q} > 0\right)$

A.2.3. *Complementarity of permits and brown goods:* The aggregate demand of the brown good decreases when the price of the permits increases. $\left(\frac{\sum \partial b_i(Y,\bar{q}_i)}{\partial P_q} < 0\right)$

A.2.4. *Relative strength of substitution effect:* The substitution effect of the ratio between the market clearing condition and the aggregate demand of the brown good is greater than the income effect generated by changes in environmental quality. $\left(\frac{\frac{\partial Q^D}{\partial P_q} - \frac{\partial Q^S}{\partial P_q}}{\frac{\sum \partial b_i}{\partial P_q}} > \frac{\frac{\partial Q^D}{\partial Y} - \frac{\partial Q^S}{\partial Y}}{\frac{\sum \partial b_i}{\partial Y}}\right)$

Assumptions A.2.1 and A.2.2 guarantee that the permits are not treated as Giffen Good. Assumption A.2.3 implies that permit buyers exhibit greater sensitivity to changes in the permit price, leading to a reduction in their consumption of brown goods that outweighs any increase in the consumption by permit sellers. Assumption A.2.4 places a bound on the extent to which variations in environmental quality affect market clearing conditions. Collectively, these four assumptions are sufficient to establish an inverse relationship between the equilibrium price and the initial allocation assigned by the regulatory agency to permit buyers.

Proposition 3.8: *If assumptions A.2.1, A.2.2, A.2.3, and A.2.4 hold, then the equilibrium price of permits decreases if any given buyer receives an additional initial permit.*

This result gives the regulatory agency a means to influence bargaining power between buyers and sellers. When buyers receive fewer initial permits, they become more active in the market, increasing demand and potentially raising the price. Conversely, a more generous allocation reduces their urgency, allowing them to adopt a more passive strategy and wait for more favorable trading conditions. The key challenge for the agency, however, lies in identifying an allocation rule that achieves a socially optimal outcome through the initial distribution of permits. The next result addresses this challenge directly:

Proposition 3.9: *Let $\bar{\boldsymbol{Q}}$ be an allocation of permits such that it induces market trading, and all consumers are either regular buyers or regular sellers. The market equilibrium with tradable permits is Pareto-Optimal if, and only if,* $P_q^* = \frac{P_b}{P_g+P_b}\left(P_g - P_b\right)$.

Starting from the social choice problem identified in (3.1), the regulator must identify a permit price that achieves two objectives: internalizing the negative externality generated by the consumption of brown goods and equalizing the marginal rates of substitution across all consumers. When all consumers are either regular sellers or regular buyers, there exists a unique price that accomplishes both goals. Specifically, the socially optimal price of permits is given by $P_q^* = \frac{\mathrm{P_b}}{\mathrm{P_g}+\mathrm{P_b}}\left(\mathrm{P_g} - \mathrm{P_b}\right)$ which simultaneously equalizes the following price ratios for buyers and sellers participating in the market.

$$\frac{P_x^d}{P_Y^d} = \frac{P_x^s}{P_Y^s}$$

If the market equilibrium with tradable permits fails to achieve the optimal price, the regulatory agency can adjust the initial allocation of permits assigned to buyers, depending on whether the prevailing equilibrium price falls below or exceeds the optimal level. By doing so, the regulator can steer the market towards the price that aligns with the social optimum.

In more general scenarios where some consumers do not participate in the market or face binding constraints, whether as restricted buyers or restricted sellers, the equilibrium with tradable permits cannot equalize the marginal rates of substitution across all individuals. As a result, the market may fail to achieve the socially optimal allocation. However, this limitation does not make the market mechanism ineffective or undesirable. Even in the presence of these imperfections, the permit system remains a valuable policy tool: it directly addresses the externality associated with the consumption of brown goods and can facilitate a redistribution of income with the objective to improve overall social welfare and equity.

Overall, the market equilibrium with tradable permits allows the regulatory agency to internalize the environmental externality and control the initial permit allocation as a redistributive mechanism. While the resulting allocation may not always coincide with the social optimum, particularly when participation or income constraints exist, the system nonetheless provides an effective and flexible policy tool to promote efficiency and equity. The existence of a well-defined optimal price enables the regulator to design and refine allocation rules that approximate the social optimum, even under less-than-ideal conditions.

4. : Conclusion

This paper presents an analysis of how socially optimal consumption of green and brown goods differs from what emerges in a market equilibrium. Achieving the social optimum requires addressing two distinct market failures: the externality caused by the consumption of brown goods and the unequal marginal rates of substitution of consumers. In some cases, a redistribution of income may be necessary to achieve a Pareto-improving outcome.

To address these failures, I propose the implementation of a permit market, drawing on the structure of the ITQs model. By restricting the consumption of brown goods to the number of permits held, the externality is directly addressed. At the same time, the initial allocation of permits has distributional implications: because each permit enables access to a lower-cost consumption option, the allocation affects the effective purchasing power of consumers. As a result, the distribution of permits not only influences overall consumption patterns but may also alter the income distribution, potentially reducing inequality.

The role of the regulatory agency is to determine the initial allocation of permits in a way that maximizes social welfare. Given that the full set of individual preferences is typically unknown, allowing consumers to trade permits provides a decentralized mechanism for discovering relative valuations. The permit price that emerges from this trading process becomes an instrument for adjusting marginal rates of substitution. For buyers, the brown good becomes more expensive, increasing the relative attractiveness of environmental quality. For sellers, the consumption of brown goods also becomes more costly, as they now face the opportunity cost of forgone permit sales.

In the social optimum, the permit price equalizes marginal rates of substitution among all consumers who participate in the market and are not constrained by their income or initial allocation. Although consumers who are inactive or constrained may retain different marginal rates of substitution, the permit market remains a valuable tool. It improves both environmental outcomes and income distribution, even if the resulting allocation does not fully coincide with the first-best optimum.

The regulator has the power to intervene in the market price by redistributing the initial permits among the groups. To achieve a higher equilibrium price, the buyers should receive fewer initial permits. Conversely, to achieve a lower equilibrium price, the buyers should receive more initial permits. It is worth noting that this reallocation of permits has implications for environmental quality. However, reaching the optimal price is a crucial step towards achieving the socially efficient level, particularly if all consumers participate as either buyers or sellers without any income or allocation constraints.

A distinctive feature of this market is that some consumers, especially sellers or non-participants, may choose to hold permits without consuming or trading them. Under typical conditions, such behavior would drive the permit price to zero. However, in this setting, permits retain value because holding them restricts others from increasing their consumption of brown goods, thereby preserving environmental quality. This intrinsic value prevents the market price from collapsing.

In conclusion, the proposed permit market offers a practical and flexible solution to the dual challenges of environmental externalities and income inequality. By reallocating resources through market prices, it supports both a more equitable distribution of consumption opportunities and an improvement in environmental quality. The effectiveness of this system ultimately depends on the regulator's ability to design and adjust the initial allocation of permits to guide the market toward the socially optimal equilibrium.

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Appendix

Proof of Proposition 3.1

Part i)

To prove that a value $P_{q_i}^{d-max}$ exists and that it is less than $P_g - P_b$, it is sufficient to show that consumer i is better off by not buying any permits if P_q exceeds a certain value. Suppose a situation where $b_i = \overline{q_\iota}$ and $g_i = \frac{1}{P_g}(w_i - P_b\overline{q_\iota})$. Consumer i would be willing to increase $\overline{q_\iota}$ in ε units ($\varepsilon > 0$) if the total change in utility is:

$$dU_i = U_X(dg_i + db_i) + U_Y(-db_i) \geq 0$$

$$= \frac{U_X}{P_g}\left(dw_i - P_b d\overline{q_\iota} + P_g d\overline{q_\iota}\right) - U_Y(d\overline{q_\iota}) \geq 0$$

Let the change in $\overline{q_\iota}$ and w_i be $d\overline{q_\iota} = \varepsilon$ and $dw_i = -P_q\varepsilon$. Then the previous expression becomes:

$$\frac{U_X}{P_g}\left(-P_q - P_b + P_g\right)\varepsilon - U_Y\varepsilon \geq 0$$

Solving for P_q gives:

$$P_q \leq P_g - P_b - P_g\frac{U_Y}{U_X}$$

$$P_q < P_g - P_b$$

Therefore, consumer i will not buy any permits when their price exceeds a value equal to $P_g - P_b - P_g\frac{U_Y}{U_X}$, which is strictly smaller than $P_g - P_b$.

Part ii)

To prove that an income level $w_i^{CD}(Y, \overline{q_\iota})$ exists, there should be a positive value of w_i for which the following relationship is true:

$$w_i - P_b\overline{q_\iota} = \left(P_b + P_q\right)q_i^d$$

Substituting (3.6) in the previous expression gives us:

$$w_i - P_b\overline{q_\iota} = (P_b + P_q)\left[\frac{{y_i^d}^{-1}\left(Y, P_X^d, P_Y^d\right) - w_i}{P_Y^d} - Y - \left(\frac{P_g - P_b}{P_Y^d}\right)\overline{q_\iota}\right]$$

$$w_i = \frac{P_b + P_q}{P_g}\left[{y_i^d}^{-1}\left(Y, P_X^d, P_Y^d\right) - P_Y^d Y\right] - P_q\overline{q_\iota}$$

If the RHS has a positive value, then it means that it is equal to the income level $w_i^B(Y, \overline{q_\iota})$. Assume by contradiction that this value is non-positive so that $w_i^B(Y, \overline{q_\iota})$ does not exist:

$$0 \geq \frac{P_b + P_q}{P_g}\left[y_i^{d^{-1}}(Y, \mathbf{P^d}) - P_Y^d Y\right] - P_q \overline{q_\iota} \qquad (B.1)$$

From the definition of $y_i^{d^{-1}}(Y, P_X^d, P_Y^d)$ we have:

$$\begin{aligned}
y_i^{d^{-1}}(Y, \mathbf{P^d}) &= w_i + P_q \overline{q_\iota} + P_Y^d Y_{-i} \\
y_i^{d^{-1}}(Y, \mathbf{P^d}) &= w_i + P_q \overline{q_\iota} + P_Y^d\left[Y + \overline{q_\iota} + q_i^d\right] \\
y_i^{d^{-1}}(Y, \mathbf{P^d}) - P_Y^d Y &= w_i + P_q \overline{q_\iota} + P_Y^d\left[\overline{q_\iota} + q_i^d\right] \\
y_i^{d^{-1}}(Y, \mathbf{P^d}) - P_Y^d Y &> (P_q + P_Y^d)\overline{q_\iota} \\
y_i^{d^{-1}}(Y, \mathbf{P^d}) - P_Y^d Y &> (P_g - P_b)\overline{q_\iota}
\end{aligned}$$

Substituting in $(B.1)$ gives:

$$\begin{aligned}
0 &> \frac{(P_b + P_q)(P_g - P_b)}{P_g}\overline{q_\iota} - P_q \overline{q_\iota} \\
0 &> \frac{P_b(P_g - P_b - P_q)}{P_g}\overline{q_\iota} > 0 \qquad \bot
\end{aligned}$$

Therefore, $\frac{P_b+P_q}{P_g}\left[y_i^{d^{-1}}(Y, P_X^d, P_Y^d) - P_Y^d Y\right] - P_q \overline{q_\iota}$ has a positive value and it is equal to the income level $w_i^B(Y, \overline{q_\iota})$.

<u>Part iii)</u>

To prove that $q_i^d(Y, \overline{q_\iota})$ is a weakly increasing function in Y, it is sufficient to check equation (3.6) is an increasing function in Y. Differentiating q_i^d with respect to Y gives us:

$$\frac{\partial q_i^d}{\partial Y} = \frac{1}{P_Y^d} \cdot \frac{\partial y_i^{d^{-1}}(Y, P_X^d, P_Y^d)}{\partial Y} - 1$$

From the inverse function theorem we have that $\frac{\partial y_i^{d^{-1}}(Y, P_X^d, P_Y^d)}{\partial Y} = \left[\frac{\partial y_i^d(m_i^d, P_X^d, P_Y^d)}{\partial m_i^d}\right]^{-1}$. From condition (3.2) we have that $\frac{\partial Y}{\partial Y_{-i}} = P_Y^d \frac{\partial y_i^d(m_i^d, P_X^d, P_Y^d)}{\partial m_i^d} < 1$. Therefore, $\frac{1}{P_Y^d} \cdot \frac{\partial y_i^{d^{-1}}(Y, P_X^d, P_Y^d)}{\partial Y} > 1$ which establishes that equation (3.6) is an increasing function in Y.

<u>Proof of </u>Proposition 3.2

To prove that a value $P_{q_i}^{s-min}$ exists and that it is strictly greater than 0, it is sufficient to show that consumer i is better off by not selling any permits if P_q does not exceed a certain value. Suppose that seller i has not entered the market yet. In this scenario, the optimal consumption of the brown good is $b_i = \frac{w_i}{P_b}$. If seller i trades ε units of a permit ($\varepsilon > 0$), then the change in income would be $dw_i = P_q\varepsilon$ and the change in the exogenous environmental quality would be $dY_{-i} = -\varepsilon$. This seller has incentives to trade away a permit as long as:

$$dU_i = U_X(db_i) + U_Y(dY_{-i} - db_i) \geq 0$$

$$U_X\left(\frac{dw_i}{P_b}\right) + U_Y\left(dY_{-i} - \frac{dw_i}{P_b}\right) \geq 0$$

$$U_X\left(\frac{P_q\varepsilon}{P_b}\right) + U_Y\left(-\varepsilon - \frac{P_q\varepsilon}{P_b}\right) \geq 0$$

$$P_q - \frac{P_b}{U_X - U_Y} \geq 0$$

Therefore, $P_{q_i}^{s-min}$ exists if $\frac{P_b}{U_X - U_Y} > 0$. From the first-order conditions of the seller problem we have that:

$$\frac{U_X - U_Y}{P_b} = \frac{U_X + \mu_g}{P_g}$$

$$\frac{U_X - U_Y}{P_b} > \frac{U_X}{P_g}$$

$$U_X - U_y > \frac{P_b}{P_g}U_X$$

$$U_X - U_y > 0$$

Therefore, consumer i will only sell a permit if $P_q \geq \frac{P_b}{U_X - U_Y} > 0$, which proves $P_{q_i}^{s-min}$ exists and is a strictly positive price.

<u>Part ii)</u>

To prove that an income level $w_i^{CS}(Y, \overline{q_i})$ exists, there should be a positive value of w_i for which $\overline{q_i} = q_i^s + b_i$. Assume by contradiction that it does not exist so that $\overline{q_i} > q_i^s + b_i, \forall w_i > 0$. From the budget constraint we get the following relationship:

$$w_i + P_q q_i^s < P_b(\overline{q_\iota} - q_i^s)$$
$$q_i^s < \frac{P_b \overline{q_\iota} - w_i}{P_b + P_q}$$
$$q_i^s < \frac{(P_X^s - P_Y^s)\overline{q_\iota} - w_i}{P_X^s}$$

Substituting this relationship in (3.10) we have:

$$\frac{(P_X^s - P_Y^s)\overline{q_\iota} - w_i}{P_X^s} > \frac{P_X^s - P_Y^s}{P_X^s}\left[\frac{y_i^{s^{-1}}(Y, \mathbf{P^s})}{P_Y^s} - Y\right] - \frac{w_i}{P_Y^s}$$
$$w_i > y_i^{s^{-1}}(Y, \mathbf{P^s}) - P_Y^S - \overline{q_\iota}$$

This last relationship implies that if $w_i \leq y_i^{s^{-1}}(Y, \mathbf{P^s}) - P_Y^S - \overline{q_\iota}$, then $\overline{q_\iota} = q_i^s + b_i$ and an income level $w_i^{CS}(Y, \overline{q_\iota})$ exists. From the definition of $y_i^{s^{-1}}(Y, \mathbf{P^s})$:

$$y_i^{s^{-1}}(Y, \mathbf{P^s}) = w_i + P_Y^s Y_{-i}$$
$$y_i^{s^{-1}}(Y, \mathbf{P^s}) = w_i + P_Y^s[Y + b_i + q_i^s]$$
$$y_i^{s^{-1}}(Y, \mathbf{P^s}) > w_i + P_Y^s Y$$
$$y_i^{s^{-1}}(Y, \mathbf{P^s}) - P_Y^s Y > w_i$$
$$y_i^{s^{-1}}(Y, \mathbf{P^s}) - P_Y^s Y > 0$$

Since $y_i^{s^{-1}}(Y, \mathbf{P^s}) - P_Y^s Y$ has a strictly positive value, then the income level $w_i^{CS}(Y, \overline{q_\iota})$ exists only if the initial allocation $\overline{q_\iota}$ is limited (i.e., $y_i^{s^{-1}}(Y, \mathbf{P^s}) - P_Y^s Y \geq \overline{q_\iota}$).

Part iii)

Note that this proof follows the same method as the one used to prove the equivalent result in Proposition 3.1, since it is sufficient to show that:

$$\frac{\partial q_i^s}{\partial Y} = \frac{1}{P_Y^s} \cdot \frac{\partial y_i^{s^{-1}}(Y, P_X^s, P_Y^s)}{\partial Y} - 1 > 0$$

Proof of Lemma 3.6

Let $P_q^{d-max}(Y^*) > P_q^{s-min}(Y^*)$. If $P_q = P_q^{d-max}(Y^*)$, then:

$$Q^D\left(P_q^{d-max}(Y^*), Y^*\right) = 0$$
$$Q^S\left(P_q^{d-max}(Y^*), Y^*\right) > 0$$
$$Q^D\left(P_q^{d-max}(Y^*), Y^*\right) < Q^S\left(P_q^{d-max}(Y^*), Y^*\right)$$

If $P_q = P_q^{s-min}(Y^*)$, then

$$Q^D\left(P_q^{s-min}(Y^*), Y^*\right) > 0$$
$$Q^S\left(P_q^{s-min}(Y^*), Y^*\right) = 0$$
$$Q^D\left(P_q^{s-min}(Y^*), Y^*\right) > Q^S\left(P_q^{s-min}(Y^*), Y^*\right)$$

Since $Q^D\left(P_q, Y^*\right)$ and $Q^S\left(P_q, Y^*\right)$ are continuous function, then by the intermediate value theorem guarantees that a price $P_q^* \in (P_q^{s-min}(Y^*), P_q^{d-max}(Y^*))$ exists such that:

$$Q^D\left(P_q^*, Y^*\right) = Q^S\left(P_q^*, Y^*\right)$$

<u>Proof of Proposition 3.7</u>

Proposition 2.1 and Lemma 3.5 guarantee that Y^* exists regardless of if there is trading or not. Suppose that at Y^* there cannot be a price P_q that clears the market. Therefore either $Q^D\left(P_q, Y^*\right) - Q^S\left(P_q, Y^*\right) < 0$ or $Q^D\left(P_q, Y^*\right) - Q^S\left(P_q, Y^*\right) > 0$. In the first case, it can only happen if $P_q > P_q^{s-min}(Y^*) > P_q^{d-max}(Y^*)$, which contradicts the assumption that $P_q^{d-max}(Y^*) > P_q^{s-min}(Y^*)$. Therefore, by Lemma 3.6*,* if $P_q^{d-max}(Y^*) > P_q^{s-min}(Y^*)$ there must exist a market-clearing price and a market equilibrium with tradable permits.

For the second case, it can only happen if $P_q^{s-min}(Y^*) > P_q^{d-max}(Y^*) > P_q$, which is also a contradiction. Finally, if $P_q^{s-min}(Y^*) \geq P_q \geq P_q^{d-max}(Y^*)$, then $Q^D\left(P_q, Y^*\right) = Q^S\left(P_q, Y^*\right) = 0$, then all $P_q \in [P_q^{d-max}(Y^*), P_q^{s-min}(Y^*)]$ would be a market clearing price.

<u>Proof of Proposition 3.8</u>

The two equations that characterize the market equilibrium with tradable permits are:

$$F^Y := Y^0 - \sum b_i - Y = 0$$
$$F^{P_q} := Q^D\left(P_q, Y\right) - Q^S\left(P_q, Y\right) = 0$$

Using the implicit function theorem, we get:

$$\begin{bmatrix} \dfrac{\partial Y}{\partial \overline{q_\iota}} \\ \dfrac{\partial P_q}{\partial \overline{q_\iota}} \end{bmatrix} = -\frac{1}{\Delta}\begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix} \begin{bmatrix} \dfrac{\partial F^Y}{\partial \overline{q_\iota}} \\ \dfrac{\partial F^{P_q}}{\partial \overline{q_\iota}} \end{bmatrix}$$

where:

$$\Delta_{11}= \frac{\partial Q^D(P_q,Y)}{\partial P_q} - \frac{\partial Q^S(P_q,Y)}{\partial P_q} < 0\ (Assumptions\ A.2.1\ and\ A.2.2)$$
$$\Delta_{12}= -\frac{\partial Q^D(P_q,Y)}{\partial Y} + \frac{\partial Q^S(P_q,Y)}{\partial Y}$$
$$\Delta_{21}= \frac{\sum b_i}{\partial P_q} < 0\ (Assumption\ A.2.3)$$
$$\Delta_{22}= -\frac{\sum b_i}{\partial Y} - 1 < 0\ \ (Lemma\ 2.4)$$
$$\Delta = \Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21} > 0\ \ (Assumption\ A.2.4)$$
$$\frac{\partial F^Y}{\partial \overline{q}_\iota} = -\frac{\sum b_i}{\partial \overline{q}_\iota} = -\frac{P_q}{P_Y^d} < 0$$
$$\frac{\partial F^{P_q}}{\partial \overline{q}_\iota} = \frac{\partial q_i^d}{\partial \overline{q}_\iota} = -\left(1 + \frac{P_q}{P_Y^d}\right) < 0$$

Therefore:

$$\frac{\partial P_q}{\partial \overline{q}_\iota} = -\frac{1}{\Delta}\left[-\frac{P_q}{P_Y^d}\Delta_{21} - \left(1 + \frac{P_q}{P_Y^d}\right)\Delta_{22}\right] < 0$$

which proves that the equilibrium price falls when a regular buyer receives one additional permit for the initial allocation.

<u>Proof of </u>Proposition 3.9

Suppose that the price $P_q \neq \frac{P_b}{P_g+P_b}(P_g - P_b)$ but that the market equilibrium with permits is socially optimal. In this case, $\frac{P_X^d}{P_Y^d} \neq \frac{P_X^s}{P_Y^s}$ which contradicts the First Fundamental Theorem of Welfare Economics.

Suppose that the price $P_q = \frac{P_b}{P_g+P_b}(P_g - P_b)$, but that the market equilibrium with permits is not socially optimal. From the method used to prove Proposition 8, it is possible to show that a redistribution of either income or permits within sellers or within buyers will cause no changes on Y^* or P_q^*. If the redistribution happens across groups, then the P_q^* will be affected, and it would contradict the First Fundamental Theorem of Welfare Economics as well.